

Quantum interference and supercurrent in multiple-barrier proximity structures

Artem V. Galaktionov and Andrei D. Zaikin

Forshchungszentrum Karlsruhe, Institut für Nanotechnologie, 76021, Karlsruhe, Germany

I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physics Institute, Leninskii pr. 53, 119991 Moscow, Russia

We analyze an interplay between the proximity effect and quantum interference of electrons in hybrid structures superconductor-normal metal-superconductor which contain several insulating barriers. We demonstrate that the dc Josephson current in these structures may change qualitatively due to quantum interference of electrons scattered at different interfaces. In junctions with few conducting channels mesoscopic fluctuations of the supercurrent are significant and its amplitude can be strongly enhanced due to resonant effects. In the many channel limit averaging over the scattering phase effectively suppresses interference effects for systems with *two* insulating barriers. In that case a standard quasiclassical approach describing scattering at interfaces by means of Zaitsev boundary conditions allows to reproduce the correct results. However, in systems with *three* or more barriers the latter approach fails even in the many channel limit. In such systems interference effects remain important in this limit as well. For short junctions these effects result in additional suppression of the Josephson critical current indicating the tendency of the system towards localization. For relatively long junctions interference effects may – on the contrary – enhance the supercurrent with respect to the case of independent barriers.

I. INTRODUCTION

In recent years there has been a great deal of activity devoted to both experimental and theoretical studies of mesoscopic superconducting-normal (SN) hybrid structures [1,2]. New important phenomena such as anomalous Meissner screening, re-entrant behavior of the conductance, nonequilibrium-driven π -junction state and many others have been discovered and thoroughly investigated. In some cases it was found that an interplay between the proximity effect and quantum interference of electrons in a normal metal play a significant role at sufficiently low temperatures. For instance, interference of electrons scattered at impurities in the normal layer may strongly enhance the Andreev conductance G_A of SN systems leading to the so-called zero-bias anomaly [3–5]: At low voltages and temperatures G_A turns out to depend linearly on the SN interface transmission D in contrast to the standard result $G_A \propto D^2$ obtained in the absence of interference effects in the N -layer.

In this paper we will address a different – although somewhat related – problem. We will analyze the dc Josephson effect in SNS systems which contain several insulating barriers. In this case electrons scattered at different barriers can interfere inside the junction. We will demonstrate that this effect may lead to qualitative modifications of the supercurrent across the junction. The most pronounced effect of quantum interference is expected in SNS systems with few conducting channels. This situation can be realized, for instance, if two superconductors are connected via a carbon nanotube [6,7]. More conventional SNS structures with many conducting channels and several insulating barriers are also of considerable interest, for instance in relation to possible applications, see e.g. Ref. [8] and further references therein. We will demonstrate that for such systems quan-

tum interference effects are also important provided there exist more than two scatterers inside the junction.

A powerful tool for theoretical studies of mesoscopic superconductivity is provided by the quasiclassical formalism of the energy-integrated Eilenberger Green functions [9] (see also Refs. [1,2,10,11] for a review). The Eilenberger equations are, on one hand, much simpler than the fully microscopic Gorkov equations and, on the other hand, allow to correctly describe the system behavior at distances much longer as compared to the Fermi wavelength $1/k_F$. Since typical length scales in superconductors (e.g. the coherence length ξ_0 or the London penetration depth) are all several orders of magnitude greater than $1/k_F$, the quasiclassical approach is usually an excellent approximation.

The quasiclassical equations cannot be applied only in the vicinity of inter-metallic interfaces and barriers where rapid changes of the system properties (at scales comparable to $1/k_F$) occur. Fortunately, in many cases this problem can be circumvented by matching the Eilenberger Green functions on both sides of the interface with the aid of the proper boundary conditions. In order to derive such conditions it is necessary to go beyond quasiclassics, however under certain assumptions the final result can be formulated only in terms of the quasiclassical Eilenberger propagators. The derivation of these boundary conditions was performed by Zaitsev [12]. Supplemented by these boundary conditions, the Eilenberger quasiclassical formalism was proven to be an extremely efficient tool for a quantitative description of numerous inhomogeneous and hybrid superconducting structures.

An important ingredient of the derivation [12] is the assumption that the boundaries are situated sufficiently far from each other, so that *interference effects* emerging from scattering at different interfaces can be totally neglected. Under this assumption one arrives at the nonlin-

ear matching conditions involving the third power of the quasiclassical propagators. These matching conditions are expressed in terms of the interface transparency coefficients for the electrons with different directions of the Fermi momenta. It is also essential that Zaitsev boundary conditions do not depend on scattering phases at the interface potentials.

Although for metallic structures containing one interface one can indeed disregard interference effects, in systems with several boundaries this is in general not anymore possible. Hence, the applicability of the nonlinear matching conditions [12] to multiple-barrier systems requires additional analysis. Some authors [13] argued that the standard quasiclassical approach can break down in a multiple-interface geometry due to the problems with the normalization of the Eilenberger functions.

In principle the above problem with boundary conditions can be avoided within the approach based on the Bogolyubov-de Gennes equations [14]. However, this approach, though frequently successful, may also be technically inconvenient in complicated situations, for instance because of a necessity to evaluate the energy eigenvalues of the system and to perform summation over the energy spectrum in the final results.

It is also possible to formulate an alternative quasiclassical approach [15] which allows to avoid the abovementioned problems. Without going into details here let us just mention that within the technique [15] one deals with the quasiclassical spinor amplitude u, v -functions which depend on one coordinate and one time only and obey linear first order equations. The Eilenberger Green functions are expressed via two linearly-independent solutions of these equations in a way that both the Eilenberger equations and the normalization conditions are automatically satisfied. The formalism [15] – just as the Eilenberger one – can be formulated both within the Matsubara and the Keldysh techniques and thus is suitable both in equilibrium and non-equilibrium situations (see, e.g. Ref. [16]) in various superconducting structures. It is also important that very simple *linear* boundary conditions for the quasiclassical amplitudes can be formulated at each of the interfaces where electron scattering takes place. The number of interfaces in the system is not restricted and the interference effects are properly taken care of. Thus it is possible to take advantage of the quasiclassical approximation and at the same time to formulate general and simple boundary conditions without making additional assumptions employed in Ref. [12].

Similar ideas have recently been put forward by Shelankov and Ozana [17]. These authors also used linear matching conditions (obtained by means of the scattering matrix approach) for the “wave functions” which factorize the two-point Green functions. The next step [17] was to construct quasiclassical one-point Green functions and formulate *nonlinear* boundary conditions for such functions which would now adequately include information

about scattering on arbitrary number of “knots”. Linear boundary conditions were also used by Brinkman and Golubov [18] in a calculation related to ours (see below).

In this paper, following Refs. [15,17,18], we will use simple linear boundary conditions in order to match the quasiclassical amplitude functions at interfaces. However, unlike in Ref. [17], we will avoid reformulating these boundary conditions as nonlinear ones for the Eilenberger Green functions. Rather we will directly express the two-point Green functions and the expectation value of the current operator in terms of the quasiclassical amplitudes. We will then apply our method to the calculation of dc Josephson currents in hybrid $SINI'S$ and $SINI'NI''S$ structures in the clean limit and for arbitrary interface transmission coefficients

The interference of the scattering events at different interfaces manifests itself in the expressions containing scattering phases ϕ at the interface potentials. For the systems with two barriers (in our case $SINI'S$ -systems) with *many transmission channels* the summation over their contributions is equivalent to effective averaging over ϕ . In this limit one can demonstrate that *after* such averaging our result is equivalent one obtained from the Eilenberger equations supplemented by the Zaitsev boundary conditions. However, in the case of more than two barriers (i.e. for $SINI'NI''S$ junctions) the dependence on the scattering phases turns out to be much more essential. In this situation the approach employing Zaitsev boundary conditions turns out to fail also in the many channel limit where quantum interference effects survive even after averaging over the scattering phases.

The paper is organized as follows. Our quasiclassical approach is outlined in Sec. II. In Sec. III we apply this approach for the analysis of the dc Josephson effect in $SINI'S$ structures with arbitrary interface transmissions. The Josephson current across $SINI'NI''S$ structures is evaluated in Sec. IV. In Sec. V we present a brief discussion and summary of our results. Some technical details of our calculation are relegated to Appendices.

II. GENERAL METHOD

A. Quasiclassical approximation

The starting point of our analysis are the microscopic Gor'kov equations [19]. In what follows we will assume that our system is uniform along the directions parallel to the interfaces (coordinates y and z). Performing the Fourier transformation of the normal G and anomalous F^+ Green function with respect to these coordinates

$$G_{\omega_n}(\mathbf{r}, \mathbf{r}') = \int \frac{d^2 \mathbf{k}_{\parallel}}{(2\pi)^2} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) e^{i\mathbf{k}_{\parallel}(\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel})}$$

we express the Gor'kov equations in the following standard form

$$\begin{pmatrix} i\omega_n - \hat{H} & \Delta(x) \\ \Delta^*(x) & i\omega_n + \hat{H}_c \end{pmatrix} \begin{pmatrix} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) \\ F_{\omega_n}^+(x, x', \mathbf{k}_{\parallel}) \end{pmatrix} = \begin{pmatrix} \delta(x - x') \\ 0 \end{pmatrix}. \quad (1)$$

Here $\omega_n = (2n + 1)\pi T$ is the Matsubara frequency, and $\Delta(x)$ is the superconducting order parameter. The Hamiltonian \hat{H} in Eq.(1) reads

$$\hat{H} = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \frac{\tilde{\mathbf{k}}_{\parallel}^2}{2m} - \epsilon_F + V(x). \quad (2)$$

Here $\tilde{\mathbf{k}}_{\parallel} = \mathbf{k}_{\parallel} - \frac{e}{c} \mathbf{A}_{\parallel}(x)$, ϵ_F is Fermi energy, the term $V(x)$ accounts for the external potentials (including the boundary potential). The Hamiltonian \hat{H}_c is obtained from \hat{H} (2) by inverting the sign of the electron charge e . The above Hamiltonians can also include the self-energy terms which, however, will not be considered below.

The quasiclassical approximation makes it possible to conveniently separate fast oscillations of the Green functions due to the factor $\exp(\pm ik_x x)$ from the envelope of these functions changing at much longer scales as compared to the atomic ones. Making use of this approximation for two-component vector $\bar{\varphi}_{\pm}(x) \exp(\pm ik_x x)$ we obtain

$$\begin{pmatrix} i\omega_n - \hat{H} & \Delta(x) \\ \Delta^*(x) & i\omega_n + \hat{H}_c \end{pmatrix} \bar{\varphi}_{\pm}(x) e^{\pm ik_x x} \simeq e^{\pm ik_x x} \begin{pmatrix} i\omega_n - \hat{H}_{\pm}^a & \Delta(x) \\ \Delta^*(x) & i\omega_n + \hat{H}_{\pm c}^a \end{pmatrix} \bar{\varphi}_{\pm}(x), \quad (3)$$

where we defined $k_x = \sqrt{k_F^2 - k_{\parallel}^2}$ and

$$\hat{H}_{\pm}^a = \mp i v_x \partial_x - \frac{e}{c} \mathbf{A}_{\parallel}(x) v_{\parallel} + \frac{e^2}{2mc^2} \mathbf{A}_{\parallel}^2(x) + \tilde{V}(x). \quad (4)$$

Here $v_x = k_x/m$, $\tilde{V}(x)$ represents a slowly varying part of the potential which *does not* include fast variations which may occur at metallic interfaces. The latter will be accounted for by the boundary conditions to be formulated below. But first let us briefly describe the general structure of the Green functions obeying eq. (1).

B. Construction of the Green functions

Consider the equation

$$\begin{pmatrix} i\omega_n - \hat{H}_{\pm}^a & \Delta(x) \\ \Delta^*(x) & i\omega_n + \hat{H}_{\pm c}^a \end{pmatrix} \bar{\varphi}_{\pm} = 0. \quad (5)$$

There exist two linearly independent solutions $\bar{\varphi}_{+}$ of eq. (5). One such solution (denoted below by $\bar{\varphi}_{+1}$) does not diverge at $x \rightarrow +\infty$, the other solution $\bar{\varphi}_{+2}$ is well-behaved at $x \rightarrow -\infty$. Similarly, two linearly independent solutions $\bar{\varphi}_{-1,2}$ do not diverge respectively at $x \rightarrow -\infty$ and $x \rightarrow +\infty$.

A particular solution of the Gor'kov equations (1) can now be sought in the following form

$$\begin{pmatrix} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) \\ F_{\omega_n}^+(x, x', \mathbf{k}_{\parallel}) \end{pmatrix} = \bar{\varphi}_{+1}(x) g_1(x') e^{ik_x(x-x')} + \bar{\varphi}_{-2}(x) g_2(x') e^{-ik_x(x-x')} \quad \text{if } x > x' \quad (6)$$

and

$$\begin{pmatrix} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) \\ F_{\omega_n}^+(x, x', \mathbf{k}_{\parallel}) \end{pmatrix} = \bar{\varphi}_{-1}(x) f_1(x') e^{-ik_x(x-x')} + \bar{\varphi}_{+2}(x) f_2(x') e^{ik_x(x-x')} \quad \text{if } x < x'. \quad (7)$$

These functions satisfy Gor'kov equations at $x \neq x'$. The functions $f_{1,2}(x)$ and $g_{1,2}(x)$ are determined with the aid of the continuity condition for the Green functions at $x = x'$ and the condition resulting from the integration of $\delta(x - x')$ in eq.(1). As a result we arrive at the linear equations

$$\begin{aligned} \bar{\varphi}_{+1}(x) g_1(x) + \bar{\varphi}_{-2}(x) g_2(x) &= \\ \bar{\varphi}_{-1}(x) f_1(x) + \bar{\varphi}_{+2}(x) f_2(x), & \\ \frac{iv_x}{2} [\bar{\varphi}_{+1}(x) g_1(x) - \bar{\varphi}_{-2}(x) g_2(x) + & \\ \bar{\varphi}_{-1}(x) f_1(x) - \bar{\varphi}_{+2}(x) f_2(x)] &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \quad (8)$$

which can be trivially resolved.

For a homogeneous superconductor in the absence of the magnetic field this procedure allows to immediately recover the well known result

$$\begin{aligned} G_{\omega_n}(x, x') &= -\frac{i}{v_x(1+\gamma^2)} \left(e^{ik_S|x-x'|} - \gamma^2 e^{-ik_S^*|x-x'|} \right), \\ F_{\omega_n}^+(x, x') &= \frac{\gamma e^{-i\chi}}{v_x(1+\gamma^2)} \left(e^{ik_S|x-x'|} + e^{-ik_S^*|x-x'|} \right), \end{aligned}$$

where χ is the phase of the pairing potential, $k_S = k_x + i\Omega_n/v_x$, $\Omega_n = \sqrt{|\Delta|^2 + \omega_n^2}$ and $\gamma = \frac{|\Delta|}{\omega_n + \Omega_n}$. Here for convenience we set $\omega_n > 0$.

In a non-homogeneous situation a general solution of the Gor'kov equations takes the form

$$\begin{aligned} \begin{pmatrix} G_{\omega_n}(x, x') \\ F_{\omega_n}^+(x, x') \end{pmatrix} &= \begin{pmatrix} G_{\omega_n}(x, x') \\ F_{\omega_n}^+(x, x') \end{pmatrix}_{part} + \\ [l_1(x') \bar{\varphi}_{+1}(x) + l_2(x') \bar{\varphi}_{+2}(x)] e^{ik_x x} &+ \\ [l_3(x') \bar{\varphi}_{-1}(x) + l_4(x') \bar{\varphi}_{-2}(x)] e^{-ik_x x}. & \end{aligned} \quad (9)$$

For systems which consist of several metallic layers the particular solution is obtained with the aid of the procedure outlined above provided both coordinates x and x' belong to the same layer. Should x and x' belong to different layers, the particular solution is zero because in that case the δ -function in eq. (1) fails. The functions $l_{1,2,3,4}(x')$ in each layer should be derived from the proper boundary conditions which we will now specify.

C. Boundary conditions

In what follows we shall assume interfaces to be non-magnetic. In this case matching of the wave functions on the left and on the right side of a potential barrier, respectively $A_1 \exp(ik_{1x}x) + B_1 \exp(-ik_{1x}x)$ and $A_2 \exp(ik_{2x}x) + B_2 \exp(-ik_{2x}x)$, is performed in a standard way (see e.g. [20]):

$$A_2 = \alpha A_1 + \beta B_1, \quad B_2 = \beta^* A_1 + \alpha^* B_1, \\ |\alpha|^2 - |\beta|^2 = \frac{k_{1x}}{k_{2x}}. \quad (10)$$

The reflection and transmission coefficients are given by

$$R = \left| \frac{\beta}{\alpha} \right|^2, \quad D = 1 - R = \frac{k_{1x}}{k_{2x}|\alpha|^2}. \quad (11)$$

For the sake of simplicity below we shall set $k_{1x} = k_{2x}$. Since typical energies of interest, such as Δ and typical Matsubara frequencies, are all much smaller than the magnitude of the interface potentials, the relationships (10) can be directly applied to the two-element columns in eq. (9). In this way we uniquely determine the Green functions of our problem.

For an illustration let us consider a metallic layer with the left and right boundaries located respectively at $x = d_1$ and $x = d_2$. We will also choose the argument x' inside this layer. As it was already explained, the particular solution of the Gor'kov equations for $x < d_1$ or $x > d_2$ is $G(x, x') = F^\pm(x, x') = 0$, while it has the form (6), (7) if the coordinate x belongs to this layer. Thus at the left boundary we get

$$\bar{\varphi}_{+2}(d_1)f_2(x')e^{-ik_x x'} + l_1(x')\bar{\varphi}_{+1}(d_1) + l_2(x')\bar{\varphi}_{+2}(d_1) = \\ \alpha_1 [l_1^L(x')\bar{\varphi}_{+1}^L(d_1) + l_2^L(x')\bar{\varphi}_{+2}^L(d_1)] + \\ \beta_1 [l_3^L(x')\bar{\varphi}_{-1}^L(d_1) + l_4^L(x')\bar{\varphi}_{-2}^L(d_1)] \quad (12)$$

and

$$\bar{\varphi}_{-1}(d_1)f_1(x')e^{ik_x x'} + l_3(x')\bar{\varphi}_{-1}(d_1) + l_4(x')\bar{\varphi}_{-2}(d_1) = \\ \beta_1^* [l_1^L(x')\bar{\varphi}_{+1}^L(d_1) + l_2^L(x')\bar{\varphi}_{+2}^L(d_1)] + \\ \alpha_1^* [l_3^L(x')\bar{\varphi}_{-1}^L(d_1) + l_4^L(x')\bar{\varphi}_{-2}^L(d_1)]. \quad (13)$$

The superscript L labels the solutions in the layer located at $x < d_1$. The above boundary conditions provide four linear equations for the functions $l(x')$ with the source term $f_{1,2}(x') \exp(\pm ik_x x')$. Similarly, with the aid of eq. (6) four boundary conditions at the right boundary $x = d_2$ can be established. Analogous procedure should be applied to other interfaces.

III. JOSEPHSON CURRENT IN *SINIS* JUNCTIONS

We shall consider *SNS* junctions composed of clean superconducting (*S*) and normal (*N*) metals. We will

assume that a thin insulating layer (*I*) can be present at both *SN* interfaces which, therefore, will be characterized by arbitrary transparencies ranging from zero to one. Specular reflection at both interfaces will be assumed. We also assume that between interfaces electrons propagate ballistically and no electron-electron or electron-phonon interactions are present in the normal metal. For simplicity we will restrict our attention to the case of identical superconducting electrodes with singlet isotropic pairing. Furthermore, we shall neglect possible suppression of the superconducting order parameter Δ in the electrodes close to the *SN* interface. This is a standard approximation which is well justified in a large number of cases. The phase of the order parameter is set to be $-\chi/2$ in the left electrode and $\chi/2$ in the right one. The thickness of the normal layer is denoted by d .

In order to evaluate the dc Josephson current across this structure we shall follow the quasiclassical approach described in the previous section. Technical details of our calculation are presented in Appendix A. As a result we arrive at the expression for the two point Green function in the normal layer. After that the current density can be calculated from the standard formula

$$J = \frac{ie}{m} T \sum_{\omega_n} \int \frac{d^2 k_{\parallel}}{(2\pi)^2} (\nabla_{x'} - \nabla_x)_{x' \rightarrow x} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}). \quad (14)$$

One can also rewrite the current in the form $J = J_+(\chi) - J_+^*(-\chi)$ where J_+ is defined by eq. (14) with positive Matsubara frequencies $\omega_n > 0$. Using (A9) and omitting terms oscillating at atomic distances we obtain

$$J_+ = 2ieT \sum_{\omega_n > 0} \int_{|k_{\parallel}| < k_F} \frac{d^2 k_{\parallel}}{(2\pi)^2} (V_1 - U_2), \quad (15)$$

or explicitly

$$J = 4eT \sin \chi \sum_{\omega_n > 0} \int_0^{k_F} \frac{k_x dk_x}{2\pi} \frac{\sin \chi}{\cos \chi + W}, \quad (16)$$

where we defined

$$W = \frac{4\sqrt{R_1 R_2}}{D_1 D_2} \frac{\Omega_n^2}{\Delta^2} \cos(2k_x d + \phi) \\ + \frac{\Omega_n^2(1 + R_1)(1 + R_2) + \omega_n^2 D_1 D_2}{D_1 D_2 \Delta^2} \cosh \frac{2\omega_n d}{v_x} \\ + \frac{2(1 - R_1 R_2)}{D_1 D_2} \frac{\Omega_n \omega_n}{\Delta^2} \sinh \frac{2\omega_n d}{v_x}. \quad (17)$$

Here $2k_x d + \phi$ is the phase of the product $\alpha_2^* \beta_2 \alpha_1^* \beta_1^*$. Eqs. (16), (17) provide a general expression for the dc Josephson current in *SINIS* structures valid for arbitrary transmissions D_1 and D_2 ranging from zero to one. This expression is the central result of this section.

We also note that the integral over k_x in eq. (16) can be rewritten as a sum over independent conducting channels

$$\frac{\mathcal{A}}{2\pi} \int_0^{k_F} k_x dk_x (\dots) \rightarrow \sum_m^N (\dots), \quad (18)$$

where \mathcal{A} is the junction cross section. In this case $D_{1,2}$ and $R_{1,2}$ may also depend on m . This would correspond to different transmissions for different conducting channels.

Finally let us point out that in the limit of symmetrical low transparent barriers $D_1 = D_2 \ll 1$ the problem was recently studied by Brinkman and Golubov [18]. In the corresponding limit their result (eq. (8) of Ref. [18]) is similar – although not fully equivalent – to our eqs. (16), (17).

A. Junctions with few conducting channels

Let us first analyze the above result for the case of one conducting channel $N = 1$. We observe that the first term in eq. (17) contains $\cos(2k_x d + \phi)$ which oscillates at distances of the order of the Fermi wavelength. Provided at least one of the barriers is highly transparent and/or (for sufficiently long junctions $d \gtrsim \xi_0$) the temperature is high $T \gg v_F/d$ this oscillating term is unimportant and can be neglected. However, at lower transmissions of both barriers and for relatively short junctions $d \lesssim v_F/T$ this term turns out to be of the same order as the other contributions to W (17). In this case the supercurrent is sensitive to the exact positions of the discrete energy levels inside the junction which can in turn vary considerably if d changes at the atomic scales $\sim 1/k_F$. Hence, one can expect sufficiently strong sample-to-sample fluctuations of the Josephson current even for junctions with nearly identical parameters.

Let us first consider the limit of relatively short *SINIS* junctions in which case we obtain

$$I = \frac{e\Delta}{2} \frac{\mathcal{T} \sin \chi}{\mathcal{D}} \tanh \left[\frac{\mathcal{D}\Delta}{2T} \right], \quad (19)$$

where we defined

$$\mathcal{D} = \sqrt{1 - \mathcal{T} \sin^2(\chi/2)} \quad (20)$$

and an effective normal transmission of the junction

$$\mathcal{T} = \frac{D_1 D_2}{1 + R_1 R_2 + 2\sqrt{R_1 R_2} \cos(2k_x d + \phi)}. \quad (21)$$

Eq. (19) has exactly the same functional form as the result derived by Haberkorn *et al.* [21] for *SIS* junctions with an arbitrary transmission of the insulating barrier.

This result is recovered from our eqs. (19), (21) if we assume e.g. $D_1 \ll D_2$ in which case the total transmission (21) reduces to $\mathcal{T} \simeq D_1$.

As we have already discussed the total transmission \mathcal{T} and, hence, the Josephson current fluctuate depending on the exact position of the bound states inside the junction. The resonant transmission is achieved for $2k_x d + \phi = \pm\pi$, in which case we get

$$\mathcal{T}_{\text{res}} = \frac{D_1 D_2}{(1 - \sqrt{R_1 R_2})^2}. \quad (22)$$

This equation demonstrates that for symmetric junctions $D_1 = D_2$ at resonance the Josephson current does not depend on the barrier transmission at all. In this case $\mathcal{T}_{\text{res}} = 1$ and our result (19) coincides with the formula derived by Kulik and Omel'yanchuk [22] for ballistic constrictions. In the limit of low transmissions $D_{1,2} \ll 1$ we recover the standard Breit-Wigner formula $\mathcal{T}_{\text{res}} = 4D_1 D_2 / (D_1 + D_2)^2$ and reproduce the result obtained by Glazman and Matveev [23] for the problem of resonant tunneling through a single Anderson impurity between two superconductors.

Note that our results (19-21) also support the conclusion reached by Beenakker [24] that the Josephson current across sufficiently short junctions has a universal form and depends only on the total scattering matrix of the weak link which can be evaluated in the normal state. Although this conclusion is certainly correct in the limit $d \rightarrow 0$, its applicability range depends significantly on the physical nature of the scattering region. From eqs. (16), (17) we observe that the result (19), (20) applies at $d \ll \xi_0$ not very close to the resonance. On the other hand, at resonance the above result is valid only under a more stringent condition $d \ll \xi_0 D_{\text{max}}$, where we define $D_{\text{max}} = \max(D_1, D_2)$.

Now let us briefly analyze the opposite limit of sufficiently long junctions $d \gg \xi_0$. Here we will restrict ourselves to the most interesting case $T = 0$. From eqs. (16), (17) we obtain

$$I = \frac{ev_x \sin \chi}{\pi d z_1} \left[\frac{\arctan \sqrt{z_2/z_1}}{\sqrt{z_2/z_1}} \right], \quad (23)$$

$$z_1 = \cos^2(\chi/2) + \frac{1}{D_1 D_2} \left(R_+ + 2\sqrt{R_1 R_2} \cos(2k_x d + \phi) \right),$$

$$z_2 = \sin^2(\chi/2) + \frac{1}{D_1 D_2} \left(R_+ - 2\sqrt{R_1 R_2} \cos(2k_x d + \phi) \right),$$

where $R_+ = R_1 + R_2$. For a fully transparent channel $D_1 = D_2 = 1$ the above expression reduces to the well known Ishii-Kulik result [25,26]

$$I = \frac{ev_x \chi}{\pi d}, \quad -\pi < \chi < \pi, \quad (24)$$

whereas if one transmission is small $D_1 \ll 1$ and $D_2 \approx 1$ we reproduce the result [27]

$$I = \frac{ev_x D_1 \sin \chi}{2d}. \quad (25)$$

Provided the transmissions of both NS -interfaces are low $D_{1,2} \ll 1$ we obtain in the off-resonant region

$$I = \frac{ev_x}{4\pi d} D_1 D_2 \sin \chi \Upsilon[2k_x d + \phi], \quad (26)$$

where $\Upsilon[x]$ is a 2π -periodic function defined as

$$\Upsilon[x] = \frac{x}{\sin x}, \quad -\pi < x < \pi. \quad (27)$$

In the vicinity of the resonance $||2k_x d + \phi| - \pi| \lesssim D_{\max}$ the above result does not hold anymore. Exactly at resonance $2k_x d + \phi = \pm\pi$ we get

$$I = \frac{ev_x \sqrt{D_1 D_2} \sin \chi}{4d \left\{ \cos^2 \frac{\chi}{2} + \frac{1}{4} \left(\sqrt{\frac{D_1}{D_2}} - \sqrt{\frac{D_2}{D_1}} \right)^2 \right\}^{1/2}}. \quad (28)$$

For a symmetric junction $D_{1,2} = D$ this formula yields

$$I = \frac{ev_x D \sin(\chi/2)}{2d}, \quad -\pi < \chi < \pi, \quad (29)$$

while in a strongly asymmetric case $D_1 \ll D_2$ we again arrive at the expression (25). This implies that at resonance the barrier with higher transmission D_2 becomes effectively transparent even if $D_2 \ll 1$. We conclude that for $D_{1,2} \ll 1$ the maximum Josephson current is proportional to the product of transmissions $D_1 D_2$ off resonance, whereas exactly at resonance it is proportional to the lowest of two transmissions D_1 or D_2 .

We observe that both for short and long $SINI'S$ junctions interference effects may enhance the Josephson effect or partially suppress it depending on the exact positions of the bound states inside the junction. We also note that in order to evaluate the supercurrent across $SINI'S$ junctions it is in general *not* sufficient to derive the transmission probability for the corresponding $NINI'N$ structure. Although the normal transmission of the above structure is given by eq. (21) for *all* values of d , the correct expression for the Josephson current can be recovered by combining eq. (21) with the results [21,24] in the limit of short junctions $d \ll D\xi_0$ only. In this case one can neglect suppression of the anomalous Green functions inside the normal layer and, hence, the information about the normal transmission turns out to be sufficient. On the contrary, for longer junctions the decay of Cooper pair amplitudes inside the N -layer cannot be anymore disregarded. In this case the supercurrent will deviate from the form (19) even though the normal transmission of the junction (21) will remain unchanged. This deviation becomes particularly pronounced for long junctions, i.e. for $d \gg \xi_0$ out of resonance and for $d \gg D\xi_0$ at resonance.

The above analysis can trivially be generalized to the case of an arbitrary number of independent conducting channels inside the junction $N > 1$. In that case the supercurrent is simply given by the sum of the contributions

from all the channels. Although all these contributions have the same form, they are in general not equal because the phase factors $2k_x d + \phi$ change randomly for different channels. Accordingly, mesoscopic fluctuations of the supercurrent should become smaller with increasing number of channels and eventually disappear in the limit of large N . In the latter limit the Josephson current is obtained by averaging over all values of the phase $2k_x d + \phi$. The corresponding results are presented below.

B. Many channel limit

Averaging over the phase factors $2k_x d + \phi$ is effectively performed by integrating over directions of the electron momentum in eq. (16). Since the term in the expression for W (17) which contains $\cos(2k_x d + \phi)$ oscillates very rapidly with changing k_x , averaging can be performed by first integrating the current (16) over the phase $2k_x d + \phi$ and then integrating the result over k_x . We obtain

$$J = \frac{2}{\pi} e k_F^2 T \sin \chi \sum_{\omega_n > 0} \int_0^1 \mu d\mu \frac{t_1(\mu) t_2(\mu)}{\mathcal{Q}^{1/2}(\chi, \mu)}. \quad (30)$$

Here and below we define $\mu = k_x/k_F$, $t_{1,2}(\mu) = D_{1,2}(\mu)/(R_{1,2}(\mu) + 1)$, $t_{\pm} = t_1 \pm t_2$ and

$$\mathcal{Q} = \left[t_1 t_2 \cos \chi + (1 + (t_1 t_2 + 1) \frac{\omega_n^2}{\Delta^2}) \cosh \frac{2\omega_n d}{\mu v_F} + t_+ \frac{\omega_n \Omega_n}{\Delta^2} \sinh \frac{2\omega_n d}{\mu v_F} \right]^2 - (1 - t_1^2)(1 - t_2^2) \frac{\Omega_n^4}{\Delta^4}. \quad (31)$$

The above equations fully determine the Josephson current in $SINI'S$ junctions in the many channel limit and at arbitrary transmissions of specularly reflecting SN -interfaces.

Let us make use of this result in order to perform a direct comparison between our analysis and the approach based on the Eilenberger equations supplemented by Zaitsev boundary conditions. The corresponding calculation within the latter approach is performed in Appendix B. It is interesting to observe that for $SINI'S$ junctions this calculation yields exactly the same result (30), (31) as obtained within our calculation after averaging over the scattering phase $2k_x d + \phi$.

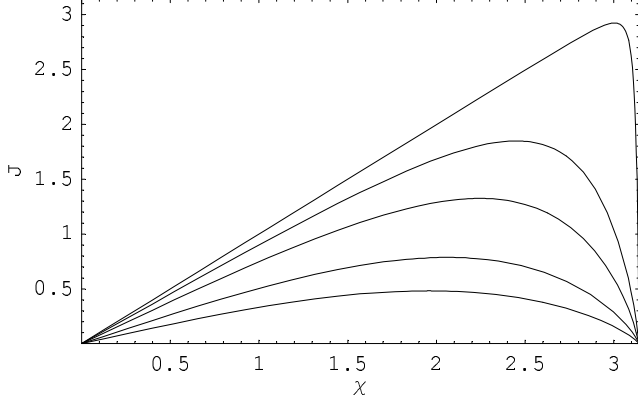


Fig. 1. The Josephson current density (33) normalized by $ek_F^2 v_F / 6\pi^2 d$ is plotted as a function of the phase difference χ . Here we assumed that the boundary is described by an effective potential $U_0 \delta(x \pm d/2)$ in which case one has $t = v_x^2 / (2U_0^2 + v_x^2)$. The dependence $J(\chi)$ was evaluated for $2U_0^2/v_F^2 = 10^{-4}, 0.03, 0.1, 0.3, 0.6$ (from top to bottom).

This observation allows to make an important conclusion concerning the applicability of the quasiclassical analysis employing Zaitsev boundary conditions for the Eilenberger propagators. The exact result for the Josephson current in *SINIS* systems, eqs. (16), (17), cannot be recovered within the latter approach because it essentially ignores interference effects arising from electron scattering on two insulating barriers. At the same time, in the limit of many conducting channels the scattering phase is effectively averaged out. In this limit Zaitsev boundary conditions turn out to correctly describe the supercurrent. It is also important to emphasize that the latter conclusion applies for the systems with not more than two barriers. Below we will analyze the supercurrent in *SNS* structures with three insulating barriers and will show that the approach based on Zaitsev boundary conditions fails to provide correct results even in the limit of many conducting channels.

But first let us present several limiting expressions for the sake of completeness. We start from the limit of a sufficiently thick junction $d \gg \xi_0$ and consider $T = 0$. In this case we find

$$J = \frac{ek_F^2 v_F \sin \chi}{2d\pi^2} \int_0^1 d\mu \mu^2 t_1 t_2 \sqrt{f_1 f_2} F(\varphi, h), \quad (32)$$

where $F(\varphi, h) = \int_0^\varphi (1 - h \sin^2 \theta)^{-1/2} d\theta$ is the incomplete elliptic integral, $\varphi = \arcsin(1/\sqrt{f_1})$, $h = f_1 + f_2 - f_1 f_2$ and

$$f_1 = \frac{1}{1 - t_1 t_2 \cos^2 \frac{\chi}{2} - \frac{1}{2} [1 - t_1 t_2 - \sqrt{(1 - t_1^2)(1 - t_2^2)}]},$$

$$f_2 = \frac{1}{1 - t_1 t_2 \sin^2 \frac{\chi}{2} - \frac{1}{2} [1 - t_1 t_2 - \sqrt{(1 - t_1^2)(1 - t_2^2)}]}.$$

For an *SINS* junction ($t_2 = 1$) the above result yields (cf. Ref. [28])

$$J = \frac{ek_F^2 v_F \sin \chi}{d\pi^2} \int_0^1 \frac{d\mu \mu^2 t_1}{\sqrt{1 - t_1^2 \cos^2 \chi}} \arctan \sqrt{\frac{1 - t_1 \cos \chi}{1 + t_1 \cos \chi}}.$$

The expression (32) also simplifies in the case of a symmetric junction $t_1 = t_2$

$$J = \frac{ek_F^2 v_F}{\pi^2 d} \int_0^1 \frac{\rho \mu^2 d\mu}{\sqrt{1 + \rho^2}} F\left(y, \frac{1}{1 + \rho^2}\right), \quad (33)$$

where

$$y = \arccos[t \cos(\chi/2)], \quad \rho(\mu, \chi) = \frac{t^2 \sin \chi}{2\sqrt{1 - t^2}}.$$

The current density J (33) is plotted in Fig. 1 as a function of the Josephson phase χ for several values of the barrier transmission. Note that in the case of small interface transparencies the limit $T \rightarrow 0$ is effectively achieved at temperatures much lower than tv_F/d .

Let us now proceed to the case of small transparencies of both interfaces $t_{1,2} \ll 1$. In this limit the expression (31) takes the form

$$\mathcal{Q} = \frac{\Omega_n^4}{\Delta^4} \left[\sinh \frac{2\omega_n d}{\mu v_F} + t_+ \frac{\omega_n}{\Omega_n} \right]^2 + \frac{\Omega_n^2}{\Delta^2} \mathcal{P}(\mu, \chi), \quad (34)$$

where

$$\mathcal{P}(\mu, \chi) = t_+^2(\mu) \cos^2(\chi/2) + t_-^2(\mu) \sin^2(\chi/2). \quad (35)$$

As we have already pointed out, the above result is not identical to one presented in eq. (13) of Ref. [18] (see also [8]). However, it is easy to see that this difference does not affect the final expression for the current in two important limits of short ($d \ll t\xi_0$) and long ($d \gg t\xi_0$) junctions. Only in the intermediate case $d \sim t\xi_0$ some deviations between our results and those of Ref. [18] are observed. This is demonstrated in Fig.2.

The case of short junctions $d \ll t\xi_0$ was already studied in Ref. [18]. Therefore here we only present the asymptotic expression for the current at $d \gg t\xi_0$

$$J = \frac{ek_F^2 v_F \sin \chi}{2\pi^2 d} \int_0^1 d\mu \mu^2 t_1 t_2 \ln(\epsilon_1/\epsilon_2), \quad (36)$$

where $\epsilon_1 = \min\{\mu v_F/d, \Delta\}$, $\epsilon_2 = \mu v_F/(4d\sqrt{\mathcal{P}})$ for $T \ll tv_F/d$ and $\epsilon_2 \simeq T$ for $tv_F/d \ll T \ll \epsilon_1$. The accuracy of the above formula is in general logarithmic, and it becomes next to logarithmic in the limits $d \ll \xi_0$ or $d \gg \xi_0$.

IV. JOSEPHSON CURRENT IN $SINI'NI''S$ JUNCTIONS

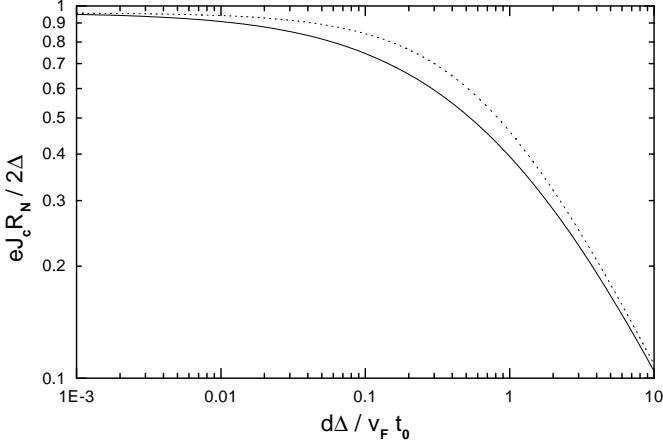


Fig. 2. The maximum Josephson current J_c of a symmetric $SINI'S$ -junction with low transparency. The angular dependence of transparency is taken to be $t(\mu) = t_0\mu^2$. The normal-state resistance of the structure is denoted by R_N . The solid line represents eqs. (30), (34) of our paper and the dashed line represents eq. (13) of Ref. [18].

Let us now consider SNS structure with three insulating barriers. As before, two of them are located at SN interfaces, and the third barrier is inside the N -layer at a distance d_1 and d_2 respectively from the left and right SN interfaces. The transmission and reflection coefficients of this intermediate barrier are denoted as D_0 and $R_0 = 1 - D_0$, whereas the left and the right barriers are characterized respectively by $D_1 = 1 - R_1$ and $D_2 = 1 - R_2$.

The supercurrent is evaluated along the same lines as it was done in Section 2 for the case of two barriers. A straightforward (although somewhat lengthy and cumbersome) procedure yields the final result for the Josephson current which is again expressed by eq. (16), where the function W now takes the form

$$W = W_+ + W_- + W_{12} \quad (37)$$

These three contributions to the W -function depend respectively on the sum of thicknesses d_1 and d_2 , their difference and on these two values separately. We find

$$W_+ = \frac{4\sqrt{R_1R_2}}{D_0D_1D_2} \frac{\Omega_n^2}{\Delta^2} \cos[2k_x(d_1 + d_2) + \phi_1 + \phi_2] + \frac{2(1-R_1R_2)}{D_0D_1D_2} \frac{\Omega_n\omega_n}{\Delta^2} \sinh \frac{2\omega_n(d_1+d_2)}{v_x} + \frac{\Omega_n^2(1+R_1)(1+R_2)+\omega_n^2D_1D_2}{D_0D_1D_2\Delta^2} \cosh \frac{2\omega_n(d_1+d_2)}{v_x}; \quad (38)$$

$$W_- = \frac{4R_0\sqrt{R_1R_2}}{D_0D_1D_2} \frac{\Omega_n^2}{\Delta^2} \cos[2k_x(d_1 - d_2) + \phi_1 - \phi_2] + \frac{2R_0(D_1-D_2)}{D_0D_1D_2} \frac{\Omega_n\omega_n}{\Delta^2} \sinh \frac{2\omega_n(d_1-d_2)}{v_x} + \frac{R_0}{D_0} \left[1 + \frac{2\Omega_n^2}{\Delta^2} \frac{R_1+R_2}{D_1D_2} \right] \cosh \frac{2\omega_n(d_1-d_2)}{v_x}; \quad (39)$$

$$W_{12} = \frac{4\sqrt{R_0R_1}}{D_0D_1} \cos[2k_xd_1 + \phi_1] \left[\frac{\Omega_n\omega_n}{\Delta^2} \sinh \frac{2\omega_nd_2}{v_x} + \frac{1+R_2}{D_2} \frac{\Omega_n^2}{\Delta^2} \cosh \frac{2\omega_nd_2}{v_x} \right] + \frac{4\sqrt{R_0R_2}}{D_0D_2} \cos[2k_xd_2 + \phi_2] \left[\frac{\Omega_n\omega_n}{\Delta^2} \sinh \frac{2\omega_nd_1}{v_x} + \frac{1+R_1}{D_1} \frac{\Omega_n^2}{\Delta^2} \cosh \frac{2\omega_nd_1}{v_x} \right]. \quad (40)$$

Here we introduced two phases

$$\begin{aligned} 2k_xd_1 + \phi_1 &= \arg \alpha_0^* \beta_0 \alpha_1^* \beta_1^*, \\ 2k_xd_2 + \phi_2 &= \arg \alpha_0^* \beta_0^* \alpha_2^* \beta_2, \end{aligned} \quad (41)$$

related to the corresponding elements of the scattering matrices for all three barriers. In contrast to the case of two barriers these phases cannot be simultaneously removed by shifting k_x . The expression for the Josephson current in $SINI'S$ junctions derived in the previous section can easily be recovered if we set $D_0 = 1 - R_0 = 1$. By setting $D_{1,2} = 1 - R_{1,2} = 0$ in the above equations we arrive at the result for the supercurrent in $SNINS$ systems derived in Ref. [27].

A. One channel limit

Let us first analyze the above general result in the limit of one conducting channel. In the limit of short junctions

$d \ll \xi_0 D_{\max}$ we again reproduce the result (19) where the total effective transmission of the normal structure with three barriers takes the form

$$\mathcal{T} = \frac{2t_1t_0t_2}{1 + t_1t_0t_2 + \mathcal{C}(\varphi_{1,2}, t_{0,1,2})}, \quad (42)$$

where

$$\begin{aligned} \mathcal{C} &= \cos \varphi_1 \sqrt{(1-t_0^2)(1-t_1^2)} + \cos \varphi_2 \sqrt{(1-t_0^2)(1-t_2^2)} \\ &+ (\cos \varphi_1 \cos \varphi_2 - t_0 \sin \varphi_1 \sin \varphi_2) \sqrt{(1-t_1^2)(1-t_2^2)}. \end{aligned} \quad (43)$$

Here we define $t_{0,1,2} = D_{0,1,2}/(1 + R_{0,1,2})$ and $\varphi_{1,2} = 2k_xd_{1,2} + \phi_{1,2}$. For later purposes let us also perform averaging of this transmission over the phases φ_1 and φ_2 . We obtain

$$\langle T \rangle = \frac{2t_1 t_0 t_2}{\sqrt{2t_1 t_0 t_2 + t_1^2 t_0^2 + t_1^2 t_2^2 + t_0^2 t_2^2 - t_1^2 t_0^2 t_2^2}}. \quad (44)$$

In particular, in the case of similar barriers with small transparencies $D_{0,1,2} \approx D \ll 1$ the average normal transmission of our structure is $\langle T \rangle \sim D^{3/2}$. Suppression of the average transmission below the value $\sim D$ is a result of destructive interference and indicates the tendency of the system towards localization. Eq. (44) follows from an explicit integration, but it can also be understood in simple terms. Consider the square $0 < \varphi_1 < 2\pi, 0 < \varphi_2 < 2\pi$. The main contribution to the average transmission comes from the resonant region $\mathcal{T} \sim 1$. In the symmetric case $t_{0,1,2} = t \ll 1$ this resonance occurs approximately along the lines $\left[\sqrt{(1 + \cos \varphi_1)(1 + \cos \varphi_2)} - t \right]^2 \sim t^3$ in two quadrants $\varphi_1, \varphi_2 < \pi$ and $\varphi_1, \varphi_2 > \pi$. In other words, the resonant region is represented by two hyperbola-like curves with characteristic widths $\sim D^{3/2}$. This dependence of the average transmission is recovered from the exact result (44).

Let us now proceed to the limit of a long junction $d_{1,2} \gg \xi_0$ and $T = 0$. In the off-resonant region we find

$$I = \frac{ev_x D_1 D_0 D_2 \sin \chi}{8\pi d_1} \mathcal{B}(\varphi_{1,2}, d_2/d_1), \quad (45)$$

where

$$\mathcal{B} = \int_0^\infty \frac{dx}{[\cosh x + \cos \varphi_1][\cosh(d_2 x/d_1) + \cos \varphi_2]}. \quad (46)$$

Evaluating this integral for $d_1 = d_2$ we get

$$J = \frac{ev_x D_1 D_0 D_2 \sin \chi}{8\pi d_1} \frac{\Upsilon[\varphi_1] - \Upsilon[\varphi_2]}{\cos \varphi_2 - \cos \varphi_1}. \quad (47)$$

This expression diverges at resonance (i.e. at $\varphi_1 \simeq \pi$ or $\varphi_2 \simeq \pi$) where it becomes inapplicable. In the resonant region $\varphi_2 \simeq \pi$ we obtain

$$I = \frac{ev_x \sqrt{D_1 D_0 D_2} \sin \chi}{4d \sqrt{2(1 + \cos \varphi_1)(\mathcal{T}^{-1} - \sin^2(\chi/2))}}. \quad (48)$$

B. Many channel junctions

As it was already discussed, in the many channel limit it is appropriate to average the current over the scattering phases. If the widths d_1 and d_2 fluctuate independently on the atomic scale, averaging over φ_1 and φ_2 can also be performed independently. If d_1 and d_2 do not change on the atomic scale but are incommensurate, independent averaging over the two phases can be performed as well. The situation is different only for strictly commensurate d_1 and d_2 in which case no independent averaging can be fulfilled.

Let us first briefly discuss the latter situation of commensurate N -layers. For simplicity we assume $d_1 = d_2$, consider a symmetric situation $D_1 = D_2 = D \ll 1$ and set the transparency of the intermediate interface to be $D_0 \gg D^2$. We will only present the result for the case of short junctions $d \ll \xi_0 D_{\max}$. We observe that the denominator in eq. (16), (37) has a resonant structure as a function of $\varphi_1 + \varphi_2$. Integrating near the resonances we obtain

$$J = \frac{1}{2\pi} e k_F^2 T \sin \chi \int_0^1 d\mu \mu D(\mu) \sum_{\omega_n > 0} \frac{\Delta^2}{\Omega_n^2 \sqrt{\mathcal{R}(\chi, \phi)}}, \quad (49)$$

where

$$\mathcal{R} = (1 + |\beta_0|^2 \sin^2 \phi)^2 \left(1 - \frac{\Delta^2 \sin^2(\chi/2)}{\Omega_n^2 (1 + |\beta_0|^2 \sin^2 \phi)} \right) \quad (50)$$

and $|\beta_0|^2(\mu) = [1 - D_0(\mu)]/D_0(\mu)$. An interesting feature of the expression (50) is a dependence of the critical Josephson current on the scattering phase $\phi = (\varphi_1 - \varphi_2)/2$. For instance, provided $|\beta_0|$ is large (the transparency of the intermediate layer is small), the critical current can vary from $\sim e k_F^2 |\Delta| D$ to $\sim e k_F^2 |\Delta| D D_0$ with ϕ varying from 0 to $\pi/2$. One should also bear in mind that ϕ may depend on μ . However, since the main contribution to the supercurrent comes predominantly from the electrons with the momenta perpendicular to the interfaces, we can estimate the current with ϕ corresponding to the forward direction. If $D_0 \ll D^2$ and $\sin^2 \phi \gg D_0$ the current is given by eq. (50). For $\phi = 0$ (or $\phi = \pi$) a different expression follows

$$J = \frac{1}{2\pi} e k_F^2 \sin \chi \Delta \tanh \frac{\Delta}{2T} \int_0^1 d\mu \mu \frac{D_0(\mu)}{D(\mu)}. \quad (51)$$

Now let us consider a more realistic situation of incommensurate d_1 and d_2 which allows for independent averaging over the scattering phases φ_1 and φ_2 . Technically this procedure amounts to evaluating the integral of the expression $1/[t + \cos x \cos(\lambda x)]$ from $x = 0$ to some large value $x = L$. At $\lambda = 1$ the result of this integration is $L/\sqrt{t(1+t)}$. However, if λ is irrational, the integral approaches the value $2LK(1/t^2)/\pi t$, where $K(h) = F(\pi/2, h)$ is the complete elliptic integral. This simple example illustrates our averaging procedure over two independent phases x and λx .

Let us assume that the transparencies of all three interfaces are small as compared to one. After averaging over φ_1 one arrives the expression which has a resonant dependence on φ_2 near $\varphi_2 = \pi$. Expanding in powers of near this resonance with $\delta\varphi_2 = \varphi_2 - \pi$ and keeping the terms proportional to $\delta\varphi_2^2$ and $\delta\varphi_2^4$ we find

$$\left\langle \frac{1}{\cos \chi + W_+ + W_- + W_{12}} \right\rangle_{\varphi_1} = \frac{\Delta^2}{2\Omega_n^2} \left\{ \frac{1}{D_1^2} + \frac{2\delta\varphi_2^2}{D_0 D_1 D_2} \left[1 - \frac{2\Delta^2 \sin^2(\chi/2)}{\Omega_n^2} \right] + \frac{\delta\varphi_2^4}{D_0^2 D_2^2} \right\}^{-1/2}.$$

Then evaluating the integral over $\delta\varphi_2$ we derive the final expression for the current

$$J = \frac{ek_F^2}{\pi^2} D_{\text{eff}} \sin \chi T \sum_{\omega_n > 0} \frac{\Delta^2}{\Omega_n^2} K \left[\frac{\Delta^2 \sin^2(\chi/2)}{\Omega_n^2} \right], \quad (52)$$

where we define the effective transmission

$$D_{\text{eff}} = \int_0^1 \mu d\mu \sqrt{D_0 D_1 D_2}. \quad (53)$$

Hence, for similar barriers we obtain the dependence $J \propto D^{3/2}$ rather than $J \propto D$ (as it would be the case for independent barriers). The latter dependence would follow from the calculation based on Zaitsev boundary conditions for the Eilenberger propagators. We observe, therefore, that quantum interference effects *decrease* the Josephson current in systems with three insulating barriers. This is essentially quantum effect which cannot be recovered from Zaitsev boundary conditions even in the multichannel limit. This effect has exactly the same origin as a quantum suppression of the average normal transmission $\langle T \rangle$ due to localization effects. Further limiting expressions for short junctions can be directly recovered from eq. (44).

We also note that the current-phase relation (52) deviates from a pure sinusoidal dependence even though all three transmissions are small $D_{0,1,2} \ll 1$. At $T = 0$ the critical Josephson current is reached at $\chi \simeq 1.7$ which is slightly higher than $\pi/2$. Although this deviation is quantitatively not very significant, it is nevertheless important as yet one more indication of quantum interference of electrons inside the junction.

Finally, let us turn to the limit of long junctions $d_{1,2} \gg \xi_0$. We again restrict ourselves to the case of low transparent interfaces. At high temperatures $T \gg v_F/2\pi d_{1,2}$ from eq. (16),(37) we get

$$J = \frac{eTk_F^2}{\pi} \frac{\Delta^2 \sin \chi}{\Delta^2 + \pi^2 T^2} \int_0^1 d\mu \mu D_0 D_1 D_2 e^{-\frac{d}{\xi(T)\mu}}, \quad (54)$$

where $d = d_1 + d_2$ and $\xi(T) = v_F/(2\pi T)$. In this case the anomalous Green function strongly decays deep in the normal layer. Hence, interference effects are not important and the interfaces can be considered as independent from each other. In the opposite limit $T \rightarrow 0$ (more precisely $T \ll Dv_F/d$), however, interference effects become important, and the current becomes proportional to $D^{5/2}$ rather than to D^3 . Explicitly, at $T \rightarrow 0$ we get

$$J = \frac{ek_F^2 v_F \sin \chi}{16\pi^2 \sqrt{d_1 d_2}} \int_0^1 d\mu \mu^2 D_1 D_2 \sqrt{D_0} \ln D_0^{-1}. \quad (55)$$

This expression is valid with the logarithmic accuracy and no distinction between $\ln D_0$, $\ln D_1$ or $\ln D_2$ should be made. We see that, in contrast to short junctions, in the limit of thick normal layers interference effects *increase* the Josephson current as compared to the case of

independent barriers. The result (55), as well as one of eqs. (52) (53) cannot be obtained from the Eilenberger approach supplemented by Zaitsev boundary conditions.

V. DISCUSSION AND CONCLUSIONS

Let us summarize our key results and observations.

In the present work we considered an interplay between the proximity effect and quantum interference of electrons in hybrid structures composed of normal metallic layers and superconductors. Quantum interference effects occur between electrons scattered at different metallic interfaces or other potential barriers and can strongly influence the supercurrent across the system.

The standard quasiclassical approach which describes scattering at interfaces by means of the nonlinear boundary conditions [12] for energy-integrated Eilenberger propagators – while very efficient in numerous other situations – is in general not suitable for the problem in question. Because of this reason we made use of an alternative quasiclassical approach which allows to investigate superconducting systems with more than one potential barrier and fully accounts for the interference effects. Within this approach scattering at boundaries is described with the aid of linear boundary conditions for quasiclassical amplitude functions. Electron propagation between boundaries is described by linear quasiclassical equations. Our approach is technically not equivalent to one based on the Bogolyubov-de Gennes equations. In particular, our method allows to explicitly construct two-point Green functions of the system and bypass such intermediate steps as finding an exact energy spectrum of the system with subsequent summation over the energy eigenvalues inevitable within the Bogolyubov-De Gennes approach. On the other hand, if needed, the full information about the energy bound states can easily be recovered within our technique by finding the poles of the Green functions in the Matsubara frequency plane.

Within our method we evaluated the dc Josephson current in SNS junctions containing two and three insulating barriers with arbitrary transmissions, respectively $SINI'S$ and $SINI'NI''S$ junctions. For the system with two barriers and few conducting channels we found strong fluctuations of the Josephson critical current depending on the exact position of the resonant level inside the junction. For short junctions $d \ll \xi_0 D$ at resonance the Josephson current does not depend on the barrier transmission D and is given by the standard Kulik-Omel'yanchuk formula [22] derived for ballistic weak links. In the limit of long SNS junctions $d \gg \xi_0$ resonant effects may also lead to strong enhancement of the supercurrent, in this case at $T \rightarrow 0$ and at resonance the Josephson current is proportional to D and not to D^2 as it would be in the absence of interference effects.

It is also interesting to observe that, while the above results for few conducting channels cannot be obtained by means of the approach employing Zaitsev boundary conditions, in the many channel limit and for junctions with two barriers the latter approach *does* allow to recover correct results. This is because the contributions sensitive to the scattering phase are effectively averaged out during summation over conducting channels or, which is the same, during averaging of the current over the directions of the Fermi velocity.

Quantum interference effects turn out to be even more important in the proximity systems which contain three insulating barriers. In this case the quasiclassical approach based on Zaitsev boundary conditions fails even in the limit of many conducting channels. In that limit the Josephson current is *decreased* for short junctions ($J \propto D^{3/2}$) as compared to the case of independent barriers ($J \propto D$). This effect is caused by destructive interference of electrons reflected from different barriers and indicates the tendency of the system towards localization. In contrast, for long *SNS* junctions with three barriers an interplay between quantum interference and proximity effect leads to enhancement of the Josephson current at $T \rightarrow 0$: We obtained the dependence $J \propto D^{5/2}$ instead of $J \propto D^3$ for independent barriers. We also discuss some further concrete results which turn out to be quite sensitive to the details of the model.

Finally, we note that in a very recent publication [29] Ozana and Shelankov analyzed the applicability of the quasiclassical technique for the case of superconducting sandwiches with several insulating barriers. For such systems they also arrived at the conclusion that in the many channel limit the standard quasiclassical scheme based on the Eilenberger equations *and* Zaitsev boundary conditions effectively breaks down in the presence of more than two reflecting interfaces. These authors argued that in such cases this scheme disregards certain classes of interfering quasiclassical paths. This conclusion [29] is similar to one reached in the present paper for *SNS* structures. We would like to point out, however, that from our point of view the failure of the above scheme is not so much due to the quasiclassical approximation and/or normalization conditions employed within the Eilenberger formalism. The problem is rather in the boundary conditions [12] which disregard interference effects which occur in the structures with several interfaces/barriers with transmissions smaller than one.

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APPENDIX A:

Let us consider an *SINI'S* system and assume that the normal metal layer is located at $-d/2 < x < d/2$. It is convenient to choose the coordinate x' within the normal layer, $-d/2 < x' < d/2$. Then a general solution of eq. (1) (decaying at $x \rightarrow -\infty$) in the left superconductor reads

$$\begin{pmatrix} G_{\omega_n}(x, x') \\ F_{\omega_n}^+(x, x') \end{pmatrix} = \begin{pmatrix} 1 \\ -ie^{i\chi/2}\gamma^{-1} \end{pmatrix} e^{\kappa x/v_x} e^{ik_x x} f(x') + \begin{pmatrix} 1 \\ ie^{i\chi/2}\gamma \end{pmatrix} e^{\kappa x} e^{-ik_x x} g(x'). \quad (\text{A1})$$

Here $\kappa = \Omega_n/v_x$. The solution in the right superconductor can be found analogously. We get

$$\begin{pmatrix} G_{\omega_n}(x, x') \\ F_{\omega_n}^+(x, x') \end{pmatrix} = \begin{pmatrix} 1 \\ ie^{-i\chi/2}\gamma \end{pmatrix} e^{-\kappa x} e^{ik_x x} n(x') + \begin{pmatrix} 1 \\ -ie^{-i\chi/2}\gamma^{-1} \end{pmatrix} e^{-\kappa x} e^{-ik_x x} r(x'). \quad (\text{A2})$$

The above solutions contain four unknown functions $f(x')$, $g(x')$, $n(x')$ and $r(x')$. These functions should be found by matching (A1), (A2) with the solution of eq. (1) in the normal layer. The latter has the form

$$\begin{pmatrix} G_{\omega_n}(x, x') \\ F_{\omega_n}^+(x, x') \end{pmatrix} = \begin{pmatrix} -\frac{i}{v_x} e^{[ik_x - (\omega_n/v_x)]|x-x'|} \\ 0 \end{pmatrix} + e^{-\omega_n x/v_x} e^{ik_x x} h(x') \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{\omega_n x/v_x} e^{-ik_x x} j(x') \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ + e^{\omega_n x/v_x} e^{ik_x x} k(x') \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{-\omega_n x/v_x} e^{-ik_x x} l(x') \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\text{A3})$$

and contain four additional unknown functions $h(x')$, $j(x')$, $k(x')$ and $l(x')$. The boundary conditions at two *NS* interfaces provide eight equations which allow to uniquely determine all the above functions and, hence, the supercurrent in *SINI'S* junctions. These equations are specified below.

Consider the left boundary. Making use of eq. (12) we find

$$\begin{pmatrix} h(x')e^{\frac{\omega_n d}{2v_x}} \\ k(x')e^{-\frac{\omega_n d}{2v_x}} \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -ie^{i\chi/2}\gamma^{-1} \end{pmatrix} e^{-\kappa d/2} f(x') + \beta_1 \begin{pmatrix} 1 \\ ie^{i\chi/2}\gamma \end{pmatrix} e^{-\kappa d/2} g(x'), \quad (\text{A4})$$

while eq. (13) yields

$$\begin{pmatrix} -\frac{i}{v_x} e^{ik_x x'} e^{-\frac{\omega_n x'}{v_x}} e^{-\frac{\omega_n d}{2v_x}} \\ 0 \end{pmatrix} + \begin{pmatrix} j(x')e^{-\frac{\omega_n d}{2v_x}} \\ l(x')e^{\frac{\omega_n d}{2v_x}} \end{pmatrix} = \alpha_1^* \begin{pmatrix} 1 \\ ie^{i\chi/2}\gamma \end{pmatrix} e^{-\kappa d/2} g(x') + \beta_1^* \begin{pmatrix} 1 \\ -ie^{i\chi/2}\gamma^{-1} \end{pmatrix} e^{-\kappa d/2} f(x'). \quad (\text{A5})$$

Similarly, applying the boundary conditions at the right interface one gets

$$\begin{pmatrix} 1 \\ ie^{-i\chi/2}\gamma \end{pmatrix} e^{-\kappa d/2} n(x') = \alpha_2 \begin{pmatrix} -\frac{i}{v_x} e^{-ik_x x'} e^{\frac{\omega_n x'}{v_x}} e^{-\frac{\omega_n d}{2v_x}} + e^{-\frac{\omega_n d}{2v_x}} h(x') \\ e^{\frac{\omega_n d}{2v_x}} k(x') \end{pmatrix} + \beta_2 \begin{pmatrix} e^{\frac{\omega_n d}{2v_x}} j(x') \\ e^{-\frac{\omega_n d}{2v_x}} l(x') \end{pmatrix}, \quad (\text{A6})$$

and

$$\begin{pmatrix} 1 \\ -ie^{-i\chi/2}\gamma^{-1} \end{pmatrix} e^{-\kappa d/2} r(x') = \alpha_2^* \begin{pmatrix} e^{\frac{\omega_n d}{2v_x}} j(x') \\ e^{-\frac{\omega_n d}{2v_x}} l(x') \end{pmatrix} + \beta_2^* \begin{pmatrix} -\frac{i}{v_x} e^{-ik_x x'} e^{\frac{\omega_n x'}{v_x}} e^{-\frac{\omega_n d}{2v_x}} + e^{-\frac{\omega_n d}{2v_x}} h(x') \\ e^{\frac{\omega_n d}{2v_x}} k(x') \end{pmatrix}. \quad (\text{A7})$$

It is easy to see that the free terms in eqs. (A4)-(A7) are

$$z_1(x') = -\frac{i}{v_x} e^{ik_x x' - (\omega_n x'/v_x)}, \quad z_2(x') = -\frac{i}{v_x} e^{-ik_x x' + (\omega_n x'/v_x)}. \quad (\text{A8})$$

Eqs. (A4)-(A7) can be trivially resolved and we arrive at the solutions for the functions $h(x')$ and $j(x')$ (which are only needed in order to evaluate the current) with the structure

$$h(x') = U_1 z_1(x') + U_2 z_2(x'), \quad j(x') = V_1 z_1(x') + V_2 z_2(x'), \quad (\text{A9})$$

where $U_{1,2}$ and $V_{1,2}$ do not depend on x' .

APPENDIX B:

Consider the Eilenberger quasiclassical propagator which has a 2×2 matrix structure in the Nambu space

$$\hat{g}(\mathbf{p}, \mathbf{R}, \omega_n) = \begin{pmatrix} g(\mathbf{p}, \mathbf{R}, \omega_n) & if(\mathbf{p}, \mathbf{R}, \omega_n) \\ -if^+(\mathbf{p}, \mathbf{R}, \omega_n) & -g(\mathbf{p}, \mathbf{R}, \omega_n) \end{pmatrix}. \quad (\text{B1})$$

Here \mathbf{p} is the electron momentum on the Fermi surface and \mathbf{R} is its coordinate. This quasiclassical propagator obeys the normalization condition $\hat{g}^2 = 1$ or, equivalently, $g^2 + ff^+ = 1$. In addition, anomalous (f, f^+) and normal (g) Green functions obey important symmetry relations

$$f^{+*}(\mathbf{p}, \mathbf{R}, \omega_n) = f(-\mathbf{p}, \mathbf{R}, \omega_n) = f(\mathbf{p}, \mathbf{R}, -\omega_n), \quad (\text{B2})$$

$$g^*(\mathbf{p}, \mathbf{R}, \omega_n) = g(-\mathbf{p}, \mathbf{R}, \omega_n) = -g(\mathbf{p}, \mathbf{R}, -\omega_n).$$

The Eilenberger equations [9] can be written in a concise matrix form as

$$i\mathbf{v}_F \nabla \hat{g} + \hat{\omega} \hat{g} - \hat{g} \hat{\omega} = 0, \quad (\text{B3})$$

$$\hat{\omega} = (i\omega_n + \frac{e}{c} \mathbf{v}_F \mathbf{A}) \hat{\sigma}_z - \hat{\Delta} + \frac{i}{2\tau} \langle \hat{g} \rangle + \frac{i}{2\tau_s} \langle \hat{\sigma}_z \hat{g} \hat{\sigma}_z \rangle,$$

where \mathbf{A} stands for the vector-potential; τ and τ_s are the elastic scattering time on nonmagnetic and paramagnetic

impurities, respectively. The angular brackets denote averaging over the Fermi surface. The matrix $\hat{\Delta}(\mathbf{R})$ incorporates the superconducting order parameter $\Delta(\mathbf{R})$, $\hat{\sigma}_z$ is the Pauli matrix

$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B4})$$

The current density $\mathbf{j}(\mathbf{R})$ is defined as follows

$$\mathbf{j}(\mathbf{R}) = -2\pi i e T N(0) \sum_m \langle \mathbf{v}_F(\mathbf{p}) g(\mathbf{p}, \mathbf{R}, \omega_n) \rangle \quad (\text{B5})$$

$N(0)$ is the density of states at the Fermi energy per one spin direction.

The Eilenberger equations (B3) should be supplemented by Zaitsev boundary conditions at metallic interfaces. These conditions have the form [12]

$$\hat{g}_{a+} = \hat{g}_{a-} = \hat{g}_a, \quad (\text{B6})$$

$$\begin{aligned} & \hat{g}_a [(1 - D(\mathbf{p}))(\hat{g}_{s+} + \hat{g}_{s-})^2 + (\hat{g}_{s+} - \hat{g}_{s-})^2] \\ & = D(\mathbf{p}) [\hat{g}_{s+} \hat{g}_{s-} - \hat{g}_{s-} \hat{g}_{s+}]. \end{aligned} \quad (\text{B7})$$

Here $\hat{g}_{s,a}(\mathbf{p}, \mathbf{R}, \omega_n) = [\hat{g}(\mathbf{p}, \mathbf{R}, \omega_n) \pm \hat{g}(\mathbf{p}_r, \mathbf{R}, \omega_n)]/2$, by \mathbf{p}_r we denote the reflected momentum. The subscripts \pm in eqs. (B6), (B7) stand for the expressions on the right

(left) side of the interface, respectively. Finally, $D(p)$ is the transparency coefficient of the boundary for the electron at the Fermi surface with the given direction of momentum. Eq. (B6) results in the current conservation at the boundary.

Consider the Josephson current in a clean *SINI'S* structure in the absence of the magnetic field. We assume that both *NS* interfaces are specularly reflecting and are perpendicular to the x -axis. In this case the quasiclassical propagator depends on p_x, x and ω_n . Making use of the symmetry relations (B2) the functions $g_{s,a}(p_x, x, \omega_n) = [g(p_x, x, \omega_n) \pm g(-p_x, x, \omega_n)]/2$ and similarly defined functions $f_{s,a}, f_{s,a}^+$ can be parametrized as follows

$$\begin{aligned} g_s &= b_{s1}, \quad f_s = b_{s2} - ib_{s3}, \quad f_s^+ = b_{s2} + ib_{s3}, \\ g_a &= ib_{a1}, \quad f_a = b_{a3} + ib_{a2}, \quad f_a^+ = -b_{a3} + ib_{a2}, \end{aligned} \quad (\text{B8})$$

The parameters b_s, b_a are real and constitute two three-dimensional vectors $\mathbf{b}_{s(a)} = (b_{s1(a1)}, b_{s2(a2)}, b_{s3(a3)})$. Combining the Eilenberger equations for $\hat{g}(\pm p_x, x, \omega_n)$, one easily finds

$$\frac{d\mathbf{b}_s}{dx} = \mathbf{M} \times \mathbf{b}_a, \quad \frac{d\mathbf{b}_a}{dx} = -\mathbf{M} \times \mathbf{b}_s. \quad (\text{B9})$$

Eqs. (B9) should be considered only for positive $p_x > 0$ and $\omega_n > 0$. The three-dimensional vector \mathbf{M} in (B9) is real and has the following components $M_1 = 2\omega/v_x$, $M_2 = (\Delta + \Delta^*)/v_x$ and $M_3 = i(\Delta - \Delta^*)/v_x$. Note that by introducing a complex vector $\mathbf{z} = \mathbf{b}_s + i\mathbf{b}_a$ one can rewrite eqs. (B9) as $d\mathbf{z}/dx = -i\mathbf{M} \times \mathbf{z}$. From the latter equation we conclude that \mathbf{z}^2 should be equal to a constant. From the normalization condition one finds $\mathbf{z}^2 = 1$, or

$$\mathbf{b}_s^2 = 1 + \mathbf{b}_a^2, \quad \mathbf{b}_s \mathbf{b}_a = 0. \quad (\text{B10})$$

The boundary conditions (B6), (B7) take the form

$$\begin{aligned} \mathbf{b}_{a-} &= \mathbf{b}_{a+} = \mathbf{b}_a, \\ [(\mathbf{b}_{s+} - \mathbf{b}_{s-})^2 + (1 - D)(\mathbf{b}_{s+} + \mathbf{b}_{s-})^2] \mathbf{b}_a &= 2D\mathbf{b}_{s+} \times \mathbf{b}_{s-} \end{aligned} \quad (\text{B11})$$

Assuming the pairing potential to be constant in the superconductors one easily recover the solution of the Eilenberger equations. For the left superconductor $x < -d/2$ we obtain

$$\mathbf{b}_s = \mathbf{e}_{M-} + \mathbf{C}_- \exp(|\mathbf{M}|x), \quad \mathbf{b}_a = \mathbf{b}_s \times \mathbf{e}_{M-}. \quad (\text{B12})$$

Here \mathbf{C}_- is an arbitrary vector perpendicular to \mathbf{M} and \mathbf{e}_{M-} is the unit vector in the direction of \mathbf{M}

$$\mathbf{e}_{M-} = \frac{1}{\sqrt{|\Delta|^2 + \omega_n^2}} \begin{pmatrix} \omega_n \\ \Delta \cos(\chi/2) \\ \Delta \sin(\chi/2) \end{pmatrix}. \quad (\text{B13})$$

Analogously, for $x > d/2$ we have

$$\mathbf{b}_s = \mathbf{e}_{M+} + \mathbf{C}_+ \exp(-|\mathbf{M}|x), \quad \mathbf{b}_a = -\mathbf{b}_s \times \mathbf{e}_{M+}, \quad (\text{B14})$$

where vector \mathbf{e}_{M+} is given by eq.(B13) with the changed sign of χ . Using these equations, from eq. (B11) one can establish the relation between \mathbf{b}_a and \mathbf{b}_s at the normal side of the interface near the left superconductor

$$\mathbf{b}_a = t_1 \mathbf{b}_s \times \mathbf{e}_{M-}. \quad (\text{B15})$$

Similarly, for the right boundary we get

$$\mathbf{b}_a = t_2 \mathbf{e}_{M+} \times \mathbf{b}_s. \quad (\text{B16})$$

With the aid of these conditions one can easily find the Josephson current in *SINI'S* junctions. What remains is to solve the Eilenberger equations (B9) in the normal metal. In the absence of external fields and impurities we have

$$\mathbf{b}_s = \begin{pmatrix} C \\ L_+ \cosh \tilde{x} + L_- \sinh \tilde{x} \\ M_+ \cosh \tilde{x} + M_- \sinh \tilde{x} \end{pmatrix}, \quad (\text{B17})$$

$$\mathbf{b}_a = \begin{pmatrix} D \\ M_+ \sinh \tilde{x} + M_- \cosh \tilde{x} \\ -L_+ \sinh \tilde{x} - L_- \cosh \tilde{x} \end{pmatrix}, \quad (\text{B18})$$

where $\tilde{x} = 2\omega_n x/v_x$ and C, D, L_{\pm}, M_{\pm} are constants determined from the normalization and boundary conditions. Finally, making use of eq. (B5) we arrive at the result (30),(31).

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